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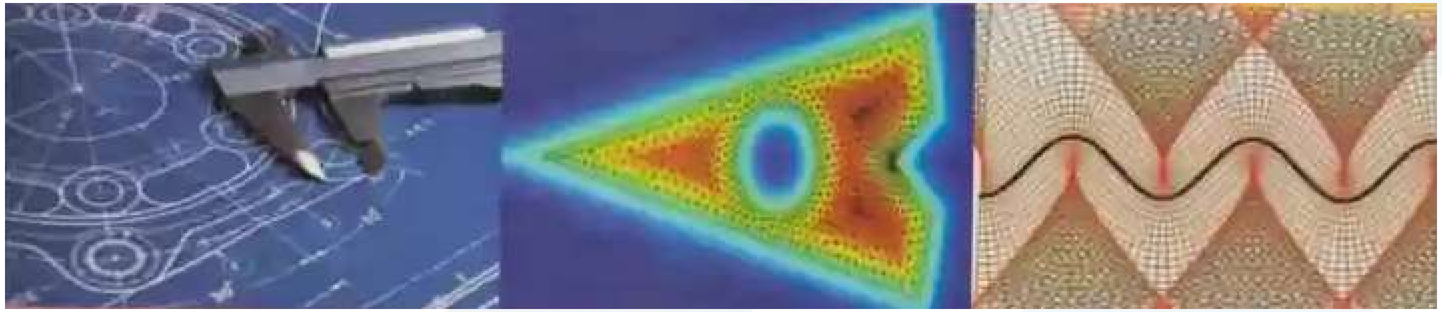
MANARA UNIVERSITY

CEDC301: Engineering Mathematics

Lecture Notes 6 & 7: Laplace Transform



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Chapter 4

Laplace Transform

1. Definition of the Laplace Transform (LT)
2. The Inverse Transform
3. Laplace Transform Properties
4. The Dirac Delta Function
5. Systems of Linear Differential Equations

1. Definition of the Laplace Transform (LT)

- Laplace transform offer simple and efficient strategies for solving many science and engineering problems, including: control systems; signal processing; mechanical networks; electrical networks and communications systems.
- The purpose of the LT is to transform ordinary differential equations (ODEs) and related initial value problems (IVP) into algebraic equations (easier to solve).

- One of the advantages of using LT to solve DE is that all initial conditions are automatically included during the process of transformation, so one does not have to find the homogeneous solutions and the particular solution separately.

- More importantly, the use of the unit step function (Heaviside function) and Dirac's delta make the Laplace transform particularly powerful for problems with inputs that have discontinuities or represent short impulses or complicated periodic functions.

- Definition: The Laplace transform of a function $f(t)$, defined for $t \geq 0$, is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} dt$$

provided that the integral converges

- The parameter s (the real part of s) must be positive and large enough to ensure that the integral converges.

- Example 1: Evaluate $\mathcal{L}\{1\}$

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \lim_{a \to \infty} \int_0^a e^{-st} dt = \frac{1}{s}, \quad s > 0$$

- Example 2: Evaluate $\mathcal{L}\{e^{at}\}$

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty e^{at}e^{-st}dt=\frac{1}{s-a},\quad s>a$$

- Example 3: Evaluate $\mathcal{L}\{\sin 2t\}$

$$\mathcal{L}\{\sin 2t\}=\int_0^\infty \sin 2te^{-st}dt=\left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty+\frac{2}{s}\int_0^\infty \cos 2te^{-st}dt$$

$$=\frac{2}{s}\int_0^\infty \cos 2te^{-st}dt,\quad s>0$$

$$=\frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty-\frac{2}{s}\int_0^\infty \sin 2te^{-st}dt\right]=\frac{2}{s^2}-\frac{4}{s^2}\mathcal{L}\{\sin 2t\}$$

$$\mathcal{L}\{\sin 2t\}=\frac{2}{s^2+4},\quad s>0$$

- Example 4: Evaluate $\mathcal{L}\{erf(a/2\sqrt{t})\}$

Error function: $erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$

$$\mathcal{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-x^2}\,dx\right]dt \qquad x = \frac{a}{2\sqrt{t}} \Rightarrow t = \frac{a^2}{4x^2}$$

$$\mathcal{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx\int_0^{a^2/4x^2} e^{-st}\,dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,\frac{1}{s}\left\{1 - exp\left(-\frac{a^2 s}{4x^2}\right)\right\}dx$$

$$= \frac{1}{s}\frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-x^2}\,dx - \int_0^\infty exp\left\{-\left(x^2 + \frac{sa^2}{4x^2}\right)\right\}dx\right]$$

$$\int_0^\infty exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}dx = \frac{1}{2}\int_0^\infty\left(1-\frac{\alpha}{x^2}\right)exp\left[-\left(x+\frac{\alpha}{x}\right)^2+2\alpha\right]dx$$

$$+\frac{1}{2}\int_0^\infty\left(1+\frac{\alpha}{x^2}\right)exp\left[-\left(x-\frac{\alpha}{x}\right)^2-2\alpha\right]dx$$

$$y=\left(x\pm\frac{\alpha}{x}\right)\Rightarrow dy=\left(1\mp\frac{\alpha}{x}\right)dx, \qquad \alpha=\frac{a\sqrt{s}}{2} \qquad \int_0^\infty e^{-y^2}dy=\frac{\sqrt{\pi}}{2}$$

$$\mathcal{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\}=\frac{1}{s}\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}e^{-a\sqrt{s}}\right]=\frac{1}{s}[1-e^{-a\sqrt{s}}]$$

Complementary error function $erfc(t)$

$$\mathcal{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\}=\frac{1}{s}e^{-a\sqrt{s}} \qquad erfc(t)=1-erf(t)=\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx$$

- Example 5: The Laplace transform of the Gaussian function: $g(t)=e^{-a^2t^2}$

$$\mathcal{L}\left\{e^{-a^2t^2}\right\}=\int_0^\infty e^{-a^2t^2}e^{-st}dt=\int_0^\infty e^{-a^2t^2-st}dt=e^{s^2/4a^2}\int_0^\infty e^{-a^2\left(1+s/2a^2\right)^2}dt$$

$$=e^{s^2/4a^2}\int_{s/2a^2}^\infty e^{-a^2u^2}du=\frac{1}{a}e^{s^2/4a^2}\int_{s/2a}^\infty e^{-v^2}dv$$

$$\mathcal{L}\left\{e^{-a^2t^2}\right\}=\frac{\sqrt{\pi}}{2a}e^{s^2/4a^2}erfc\left(\frac{s}{2a}\right)$$

$\mathcal{L}\{t^a\}$ when $a>-1$: the gamma function

$$\mathcal{L}\left\{t^a\right\}=\int_0^\infty t^a e^{-st}dt=\frac{1}{s^{a+1}}\int_0^\infty x^a e^{-x}dx=\frac{\Gamma(a+1)}{s^{a+1}},\qquad s>0$$

$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx,\ a>0$ $\Gamma(a)$ is the so-called gamma function

It can be shown that the gamma function satisfies the relation: $\Gamma(a+1)=a\,\Gamma(a)$

When a is a positive integer: $\mathcal{L}\left\{t^n\right\} = \dfrac{\Gamma(n+1)}{s^{a+1}} = \dfrac{n!}{s^{n+1}}$

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$$

$$\frac{\Gamma(x)}{s^x} = \mathcal{L}\left\{t^{x-1}\right\} = \int_0^\infty t^{x-1}e^{-st}dt, \qquad s > 0$$

$$\Gamma(x)\sum_{s=1}^{\infty}\frac{1}{s^x} = \int_0^\infty t^{x-1}\sum_{s=1}^{\infty}e^{-st}dt = \int_0^\infty t^{x-1}\frac{e^{-t}}{1-e^{-t}}dt = \int_0^\infty \frac{t^{x-1}}{e^t - 1}dt$$

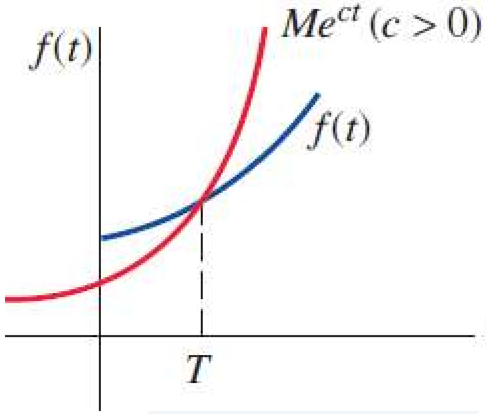
$$\zeta(x) = \sum_{s=1}^{\infty}\frac{1}{s^x} = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t - 1}dt$$

$\zeta(x)$ is the so-called Riemann zeta function

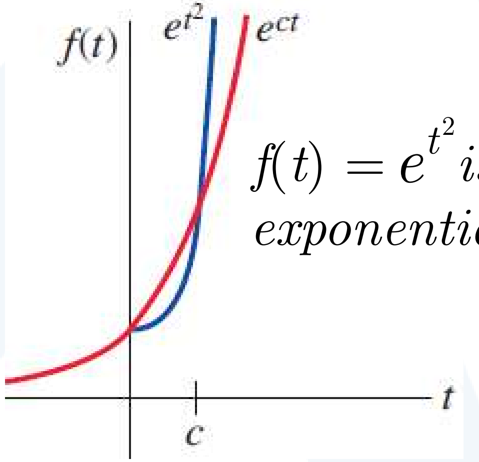
$$\zeta(2) = \sum_{s=1}^{\infty}\frac{1}{s^2} = \frac{\pi^2}{6} = \frac{1}{\Gamma(2)}\int_0^\infty \frac{t}{e^t - 1}dt = \int_0^\infty \frac{t}{e^t - 1}dt$$

Existence of Laplace Transforms

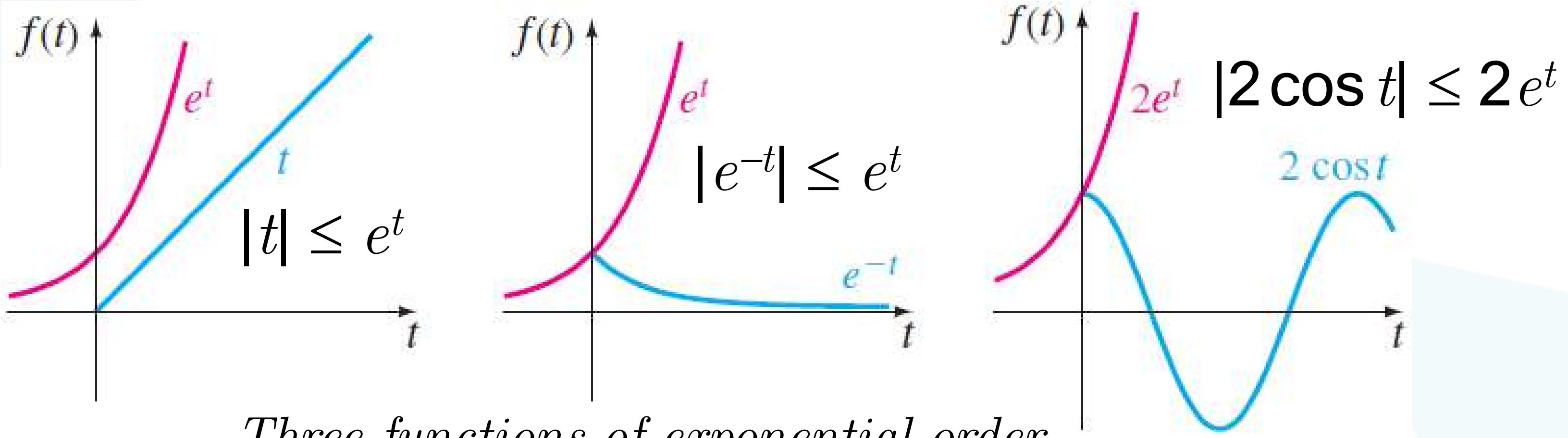
- **Definition:** A function f is said to be of exponential order if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.
- If f is an increasing function, then the condition $|f(t)| \le Me^{ct}$ for all $t > T$, simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}, where c is a positive constant.



Function f is of exponential order



$f(t) = e^{t^2}$ is not of exponential order



Three functions of exponential order

- **Theorem 1 (Sufficient Conditions for Existence):** If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.
- **Note:** The function $1/\sqrt{t}$ is not of exponential order, because of its behavior at $t = 0$. However, we show in Example 5 that its Laplace Transform exists for all $s > 0$ ($\mathcal{L}\{1/\sqrt{t}\} = \sqrt{\pi/s}$). Thus Theorem 1 provides sufficient but not necessary conditions for the existence of the LT.

Linearity of the Laplace Transform

- **Theorem 2 (Linearity of the LT):** The LT is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose LT exist and any constants a and b, the Laplace transform of $af + bg$ exists, and $\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

- **Example 6:** Laplace transform of $\sin$ and $\cos$

$$\sin \omega t = \tfrac{1}{2i}\left[e^{i\omega t} - e^{-i\omega t}\right] \Rightarrow \mathcal{L}\{\sin \omega t\} = \tfrac{1}{2i}\left[\mathcal{L}\{e^{i\omega t}\} - \mathcal{L}\{e^{-i\omega t}\}\right]$$

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2i}\left[\frac{1}{s-\omega i} - \frac{1}{s+\omega i}\right] = \frac{\omega}{s^2+\omega^2}, \qquad s > 0$$

$$\cos \omega t = \tfrac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right] \Rightarrow \mathcal{L}\{\cos \omega t\} = \tfrac{1}{2}\left[\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}\right]$$

$$\mathcal{L}\{\cos \omega t\} = \frac{1}{2}\left[\frac{1}{s-\omega i} + \frac{1}{s+\omega i}\right] = \frac{s}{s^2+\omega^2}, \qquad s > 0$$

- Theorem 3 (Behavior of $F(s)$ as $s \to \infty$): If a function $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order with c specified in the definition and $\mathcal{L}\{f(t)\} = F(s)$, then $\lim_{s \to \infty} F(s) = 0$.

Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$	
1	1	$1/s$	$s > 0$
2	t^n, $n = 0, 1, 2, \ldots$	$n!/s^{n+1}$	$s > 0$
3	t^a, $a > 0$	$\Gamma(a+1)/s^{a+1}$	$s > 0$
4	e^{at}	$1/(s-a)$	$s > a$
5	$t^n e^{at}$	$n!/(s-a)^{n+1}$	$s > a$

	$f(t)$	$\mathcal{L}(f)$	
6	$\cos \omega t$	$s/(s^2+\omega^2)$	$s>0$
7	$\sin \omega t$	$\omega/(s^2+\omega^2)$	$s>0$
8	$\cosh at$	$s/(s^2-a^2)$	$s>\lvert a\rvert$
9	$\sinh at$	$a/(s^2-a^2)$	$s>\lvert a\rvert$
10	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$s>a$
11	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$	$s>a$

2. The Inverse Transform

Inverse Transforms

- If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$. For example:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, \quad \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

- Theorem 4 (Inversion formula): The inversion formula for the Laplace transform is

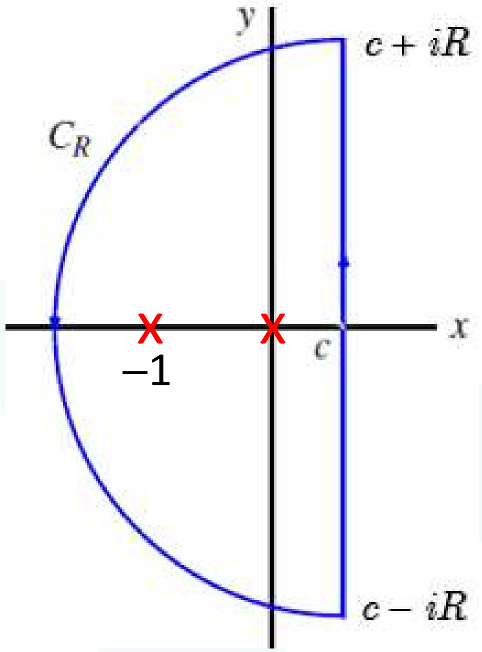
$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$$

where we will choose c so that all of the singularities of $F(s)$ have $Re\{s\} < c$.

- Example 7: Find the inverse LT of $F(s) = \dfrac{1}{s(s+1)}$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s(s+1)}\,ds$$

If $t \geq 0$, we can close the contour (vertical line to the right of the singularities) to the left. The contribution of the integral around the semicircle vanishes as its radius $R \to \infty$, giving:



$$Res\,(F(s), 0) = 1 \qquad Res\,(F(s), -1) = -e^{-t}$$

$$f(t) = \frac{1}{2\pi i} 2\pi i [Res\,(F(s), 0) + Res\,(F(s), -1)] = 1 - e^{-t}$$

- Note: Partial fractions play an important role in finding inverse Laplace transforms.

- Example 8: Partial Fractions and Linearity

Evaluate $\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1}+\frac{25/6}{s-2}+\frac{1/30}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

- Example 9: Evaluate

$$\mathcal{L}^{-1}\left\{\frac{2}{4+(s-1)^2}\right\}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{4+(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} = e^t sin\, 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\} = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\} = \frac{1}{\sqrt{2}} e^{-t} sin\, \sqrt{2}t$$

3. Laplace Transform Properties

Transforms of Derivatives

- Theorem 5 (Transform of a Derivative): If f, f', ..., $f^{(n-1)}$ are continuous on [0, ∞) and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on [0, ∞), then: $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

  where $F(s) = \mathcal{L}\{f(t)\}$.

In particular,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Solving Linear ODEs

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = g(t), \quad y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)}(0) = y_{n-1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants.

$$a_n \mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1}\mathcal{L}\left\{\frac{d^{n-1}y}{dt^{n-1}}\right\} + \cdots + a_0\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$a_n[s^nY(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)]$$
$$+ a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)] + \cdots + a_0Y(s) = G(s)$$

where $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$

- Note: The LT of a linear DE with constant coefficients becomes an algebraic equation in $Y(s)$.

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}, \qquad P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0, \qquad \deg Q(s) \le n-1$$

$Q(s)$ is a polynomial consisting of the various products of a_i, $i = 1, \ldots, n$, and the initial conditions $y_0, y_1, \ldots, y_{n-1}$.

- Example 10: Solving a First-Order IVP

Use the LT to solve the initial-value problem $\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\}$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \Rightarrow Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

- Example 11: Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

$$y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

- The Laplace transform is well adapted to linear dynamical systems

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + \mathcal{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\} = y_0(t) + y_1(t)$$

- If the input is $g(t) = 0$, then the solution of the problem is $y_0(t) = \mathcal{L}^{-1}\{Q(s)/P(s)\}$. This solution is called the zero-input response of the system.
- The function $y_1(t) = \mathcal{L}^{-1}\{G(s)/P(s)\}$ is the output due to the input $g(t)$ with zero initial conditions), which is called the zero-state response of the system.
- In example 11: $y_0(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)(s-2)}\right\} = -3e^{t} + 4e^{2t}$

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s+4)}\right\} = -\frac{1}{5}e^{t} + \frac{1}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Translation on the s-axis

- **Theorem 6 (First Translation Theorem):** If $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$ and a is any real number, then:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \text{or} \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

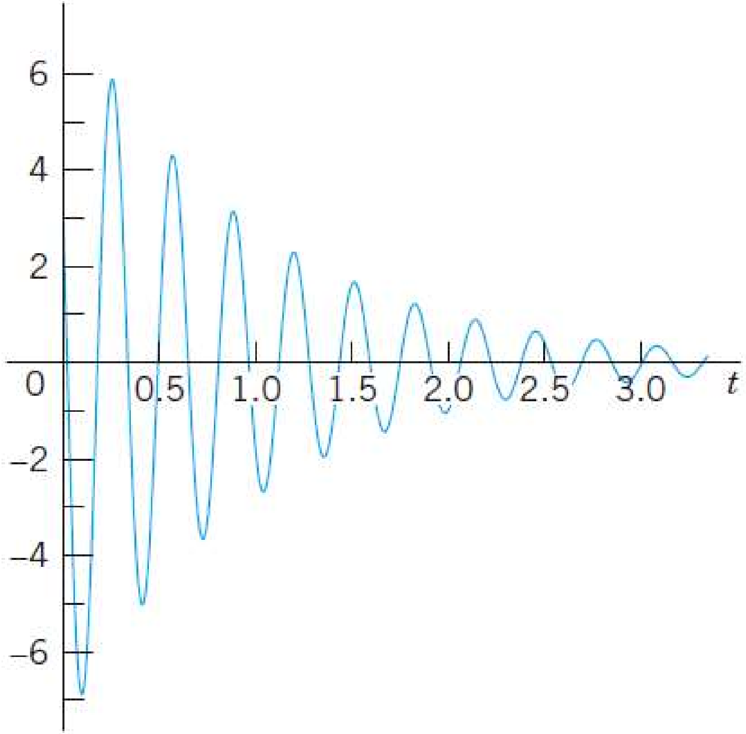
for $s > a + c$.

- **Example 12:** Damped Vibrations

Find the inverse of the transform $\mathcal{L}\{f(t)\} = \dfrac{3s-137}{s^2+2s+401}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}$$

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$$



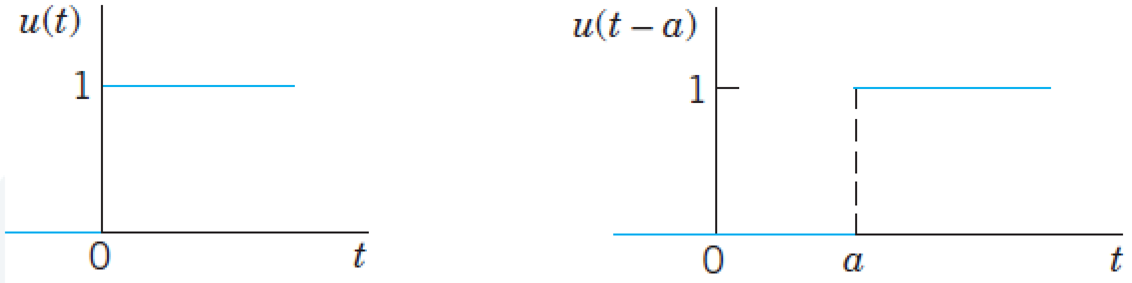
- We shall now reach the point where the Laplace transform shows its real power in applications. We shall introduce two auxiliary functions, the unit step function or Heaviside function $u(t-a)$ and Dirac's delta $\delta(t-a)$.

- These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant.

Unit Step Function

- Definition: The unit step function $u(t-a)$ is defined to be

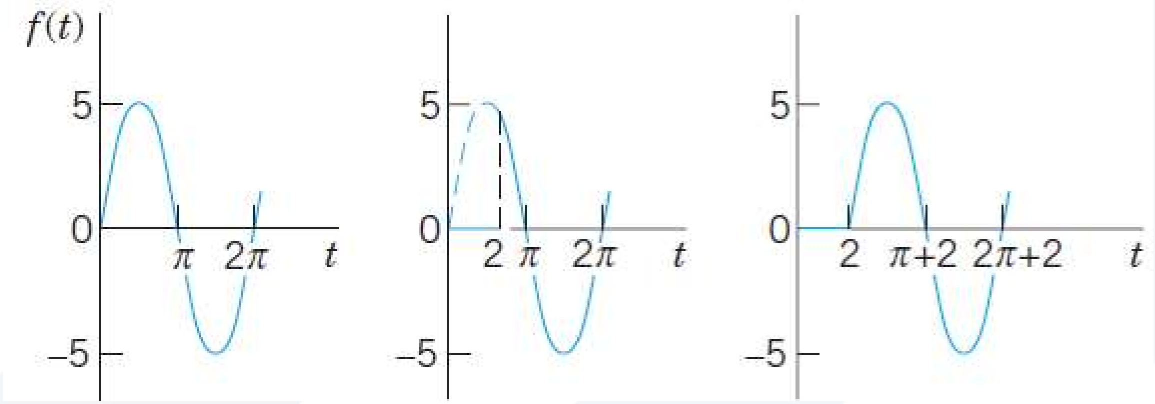
$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}, \qquad a\geq 0$$



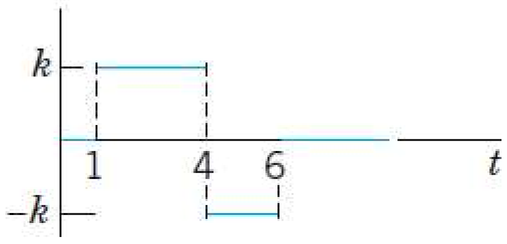
$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}dt = \int_a^\infty e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}, \quad s > 0$$

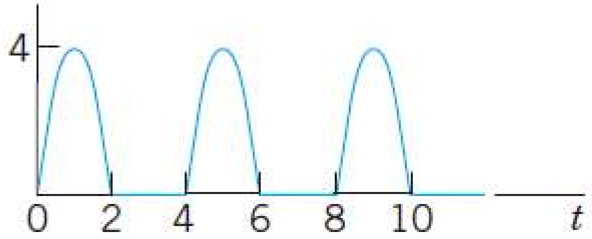
- Note: Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.



$f(t) = 5\sin t$ $\quad$ $f(t)u(t-2)$ $\quad$ $f(t-2)u(t-2)$



$k[u(t-1)-2u(t-4)+u(t-6)]$



$4\sin(\frac{1}{2}\pi t)[u(t)-u(t-2)+u(t-4)-u(t-6)+\cdots]$

Translation on the t-axis

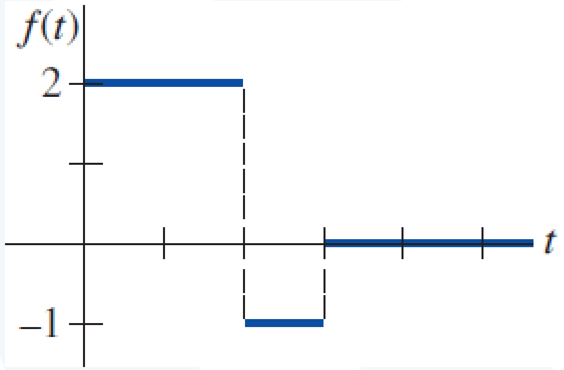
- **Theorem 7 (Second Translation Theorem):** If $\mathcal{L}\{f(t)\}=F(s)$ and $a>0$, then

$$\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s) \quad \text{or} \quad f(t-a)u(t-a)=\mathcal{L}^{-1}\{e^{-as}F(s)\}$$

for $s>c$.

- **Example 13:** Second Translation Theorem

Find the Laplace transform of the function f whose graph is given in the figure beside.



$$f(t) = 2 - 3u(t-2) + u(t-3)$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{u(t-2)\} + \mathcal{L}\{u(t-3)\}$$

$$= \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

- Example 14: An Initial-Value Problem

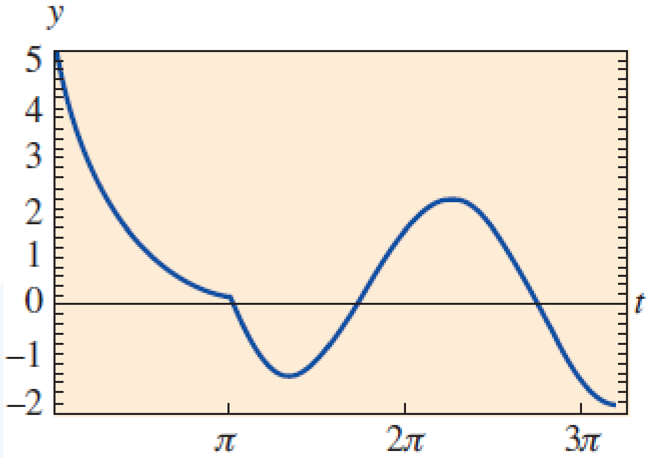
Solve $y' + y = f(t)$, $y(0) = 5$, where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t\, u(t-\pi)\} = -3\mathcal{L}\{\cos(t-\pi)\, u(t-\pi)\}$$

$$sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s} \Rightarrow Y(s) = \frac{5}{s+1} - \frac{3s}{(s+1)(s^2+1)}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

$$y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}u(t-\pi) - \frac{3}{2}\sin(t-\pi)u(t-\pi) - \frac{3}{2}\cos(t-\pi)u(t-\pi)$$
$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]u(t-\pi)$$
$$= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \sin t + \cos t], & t \ge \pi \end{cases}$$



Derivatives of Transforms

- Theorem 8 (Derivatives of Transforms): If the function $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order with c specified in the definition and $\mathcal{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, \ldots$, then:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

- Example 15: Evaluate $\mathcal{L}\{t \sin kt\}$

$$\mathcal{L}\{t\sin kt\} = -\frac{d}{ds}\mathcal{L}\{\sin kt\} = -\frac{d}{ds}\frac{k}{s^2+k^2} = \frac{2ks}{(s^2+k^2)^2}$$

- Example 16: An Initial-Value Problem Solve $ty'' + y' = t^3 + 1,\ y(0) = 0,\ y'(0) = 1$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) = -s^2Y'(s) - 2sY(s) + y(0)$$

$$-s^2Y'(s) - 2sY(s) + y(0) + sY(s) - y(0) = \frac{6}{s^4} + \frac{1}{s}$$

$$sY'(s) + Y(s) = -\frac{6}{s^5} - \frac{1}{s^2} \Rightarrow \frac{d}{ds}[sY(s)] = -\frac{6}{s^5} - \frac{1}{s^2} \Rightarrow Y(s) = \frac{3}{2}\frac{1}{s^5} + \frac{1}{s^2} + \frac{c}{s}$$

$$y(t) = \frac{1}{16}t^4 + t + c \qquad y(0) = 0 \Rightarrow y(t) = \frac{1}{16}t^4 + t$$

Convolution

If functions f and g are piecewise continuous on the interval [0, ∞), then the convolution of f and g, denoted by the symbol $f * g$, is a function defined by the integral:

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

- Example 17: Evaluate (a) $e^t * \sin t$ (b) $\mathcal{L}\{e^t * \sin t\}$ (c) $\sin t * \sin t$

(a) $e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)d\tau = \frac{1}{2}(-\sin t - \cos t + e^t)$

(b) $\mathcal{L}\{e^t * \sin t\} = -\frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s-1} = \frac{1}{(s-1)(s^2+1)}$

(c) $\sin t * \sin t = \int_0^t \sin(\tau)\sin(t-\tau)d\tau = \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos(t)]d\tau$

$= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$

Properties of Convolution

- Is commutative. For any two functions f and g, $f * g = g * f$.
- Is associative. For any functions f, g, and h, $(f * g) * h = f * (g * h)$.
- Is distributive with respect to addition. For any functions f, g, and h, $f * (g + h) = f * g + f * h$.

- Theorem 9 (Convolution Theorem): If the functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in the definition and $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$, then for $s > c$:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

From Example 17:

$$\mathcal{L}\{e^t * \sin t\} = \mathcal{L}\{e^t\}\mathcal{L}\{\sin t\} = \frac{1}{s-1}\frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Transform of an Integral

- When $g(t) = 1$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

- Example 18: An Integral Equation

  Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau$

$$\int_0^t f(\tau)e^{t-\tau}d\tau = f(t) * e^t$$

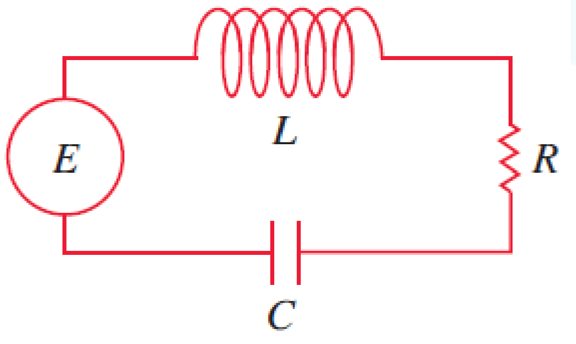
$$F(s) = 3\frac{2}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1} \Rightarrow F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

Laplace Transform in Circuit Analysis

Series Circuits

- The voltage drops across an inductor, resistor, and capacitor are, respectively,



$$v_L = L\frac{di(t)}{dt}, \quad v_R = Ri(t), \quad v_C = \frac{1}{C}\int_0^t i(\tau)d\tau$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

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- Example 19: An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit when $L = 0.1$ H, $R = 2\ \Omega$, $C = 0.1$ F, $i(0) = 0$, and the impressed voltage is $E(t) = 120t - 120t\,u(t-1)$.

$$0.1\frac{di(t)}{dt} + 2i(t) + 10\int_0^t i(\tau)\,d\tau = 120t - 120t\,u(t-1)$$

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right]$$

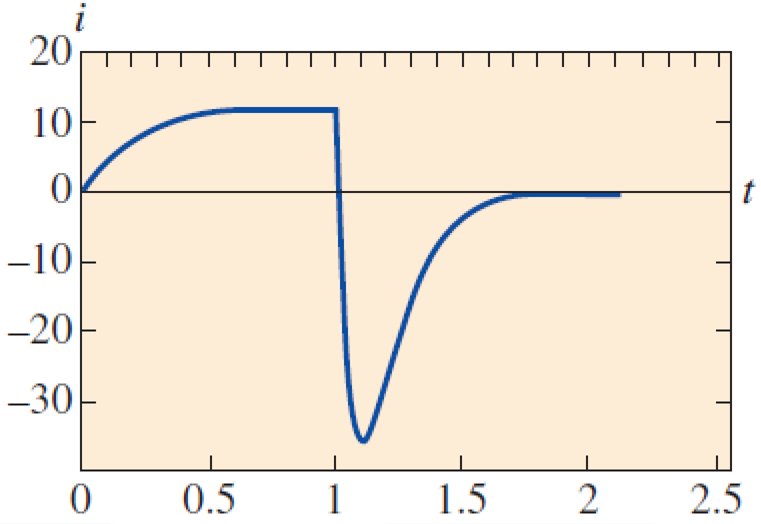
$$I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right]$$

$$I(s) = 1200\left[\frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s}\right.$$

$$+\frac{1/100}{s+10}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\Bigg]$$

$$i(t)=12[1-u(t-1)]-12[e^{-10t}-e^{-10(t-1)}u(t-1)]$$

$$-120te^{-10t}-1080(t-1)e^{-10(t-1)}u(t-1)$$



Transform of a Periodic Function

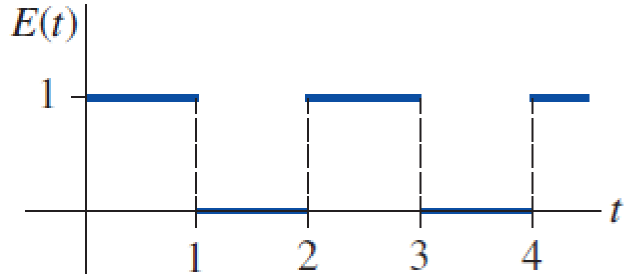
- **Theorem 10 (Transform of a Periodic Function):** If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_{0^-}^{T} e^{-st} f(t)\, dt$$

- **Example 20:** Transform of a Periodic Function

Find the Laplace transform of the periodic function shown in the figure above

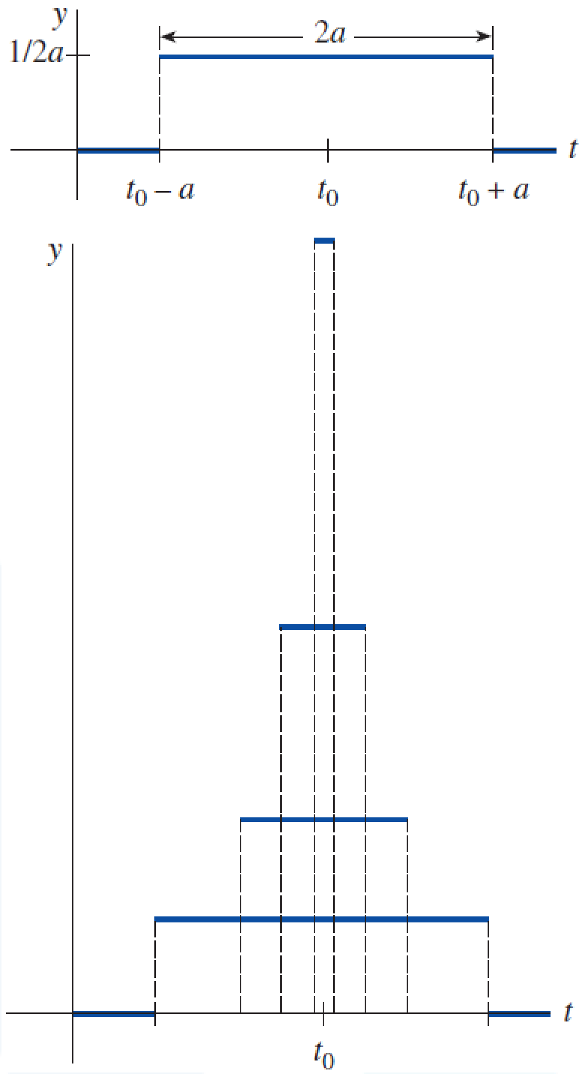
The function $E(t)$ is called a square wave and has period $T = 2$. For $0 \le t < 2$, $E(t)$ can be defined by:



$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\int_{0^-}^{2} e^{-st}E(t)\,dt = \frac{1}{1-e^{-2s}}\int_{0^-}^{1} e^{-sT}\,dt$$

$$= \frac{1}{1-e^{-2s}}\,\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$



4. The Dirac Delta Function

- We shall see that there does indeed exist a function, or more precisely a generalized function, whose Laplace transform is $F(s) = 1$.
- Mechanical systems are often acted on by an external force (or emf in an electrical circuit) of large magnitude that acts only for a very short period of time.

- The graph of the piecewise-defined function

$$\delta_a(t-t_0)=\begin{cases}0, & 0\le t<t_0-a\\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a\\ 0, & t\ge t_0+a\end{cases}$$

$a>0,\ t_0>0$, shown before could serve as a model for such a force.

- The function $\delta_a(t-t_0)$ is called a unit impulse since it possesses the integration property:

$$\int_0^{\infty}\delta_a(t-t_0)\,dt=1$$

- The Dirac Delta Function $\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$

$$\delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-t_0)\,dt=1$$

$$\int_0^\infty f(t)\delta(t-t_0)\,dt=f(t_0)$$ sifting property

- **Theorem 11 (Transform of Dirac Delta Function):** For $t_0>0$,

$$\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}, \qquad \mathcal{L}\{\delta(t)\}=1$$

- **Example 21:** Two Initial-Value Problems

  Solve $y'' + y = 4\,\delta(t-2\pi)$ subject to

  (a) $y(0)=1$, $y'(0)=0$ (b) $y(0)=0$, $y'(0)=0$.

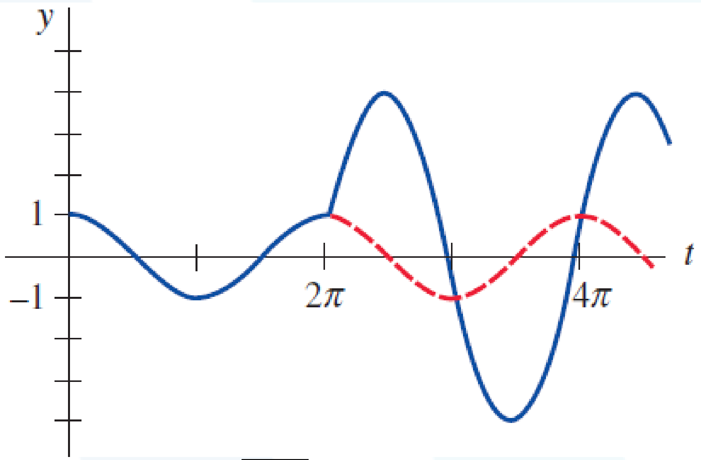
  The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible.

At $t = 2\pi$ the mass is given a sharp blow. In part (a) the mass is released from rest 1 unit below the equilibrium position. In part (b) the mass is at rest in the equilibrium position.

(a) $s^2Y(s) - s + Y(s) = 4e^{-2\pi s} \Rightarrow Y(s) = \dfrac{s}{s^2+1} + \dfrac{4e^{-2\pi s}}{s^2+1}$

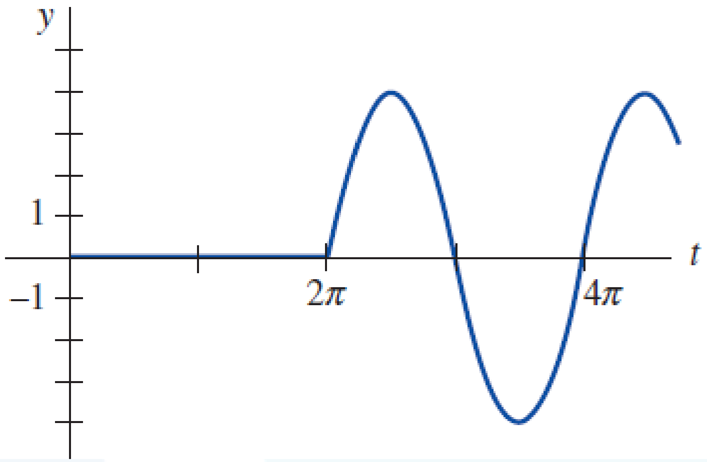
$$y(t) = \cos t + 4\sin(t - 2\pi)u(t - 2\pi)$$

$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$$



The mass is exhibiting simple harmonic motion until it is struck at $t = 2\pi$. The influence of the unit impulse is to increase the amplitude of vibration to $\sqrt{17}$ for $t > 2\pi$.

(b) $Y(s) = \dfrac{4e^{-2\pi s}}{s^2+1} \Rightarrow y(t) = 4\sin(t-2\pi)u(t-2\pi)$



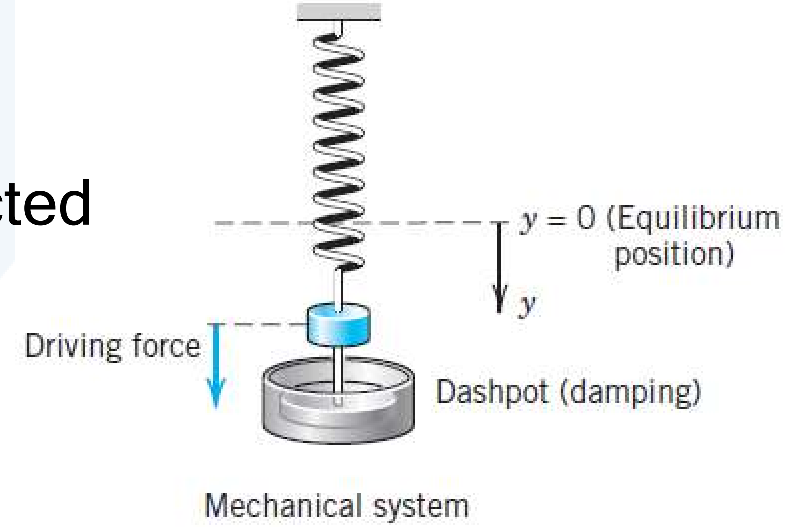
$$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 4\sin t, & t \ge 2\pi \end{cases}$$

The mass exhibits no motion until it is struck at $t = 2\pi$.

- Example 22: Damped Forced Vibrations

  Solve the IVP for a damped mass-spring system acted upon by a sinusoidal force for some time interval.

  $y'' + 2y' + 2y = r(t)$, $r(t) = 10\sin 2t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 1$, $y'(0) = -5$.



$$(s^2Y(s) - s + 5) + 2(sY(s) - 1) + 2Y(s) = 10\frac{2}{s^2+4}(1 - e^{-\pi s})$$

$$Y(s) = \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-3}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4\sin t)$$

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{-2s-2}{s^2+4} + \frac{2(s-1)+6-2}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{(s^2+4)(s^2+2s+2)}\right\} = -2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t)$$

$$y(t) = e^{-t}(\cos t - 4\sin t) - 2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t)$$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t, \qquad 0 < t < \pi$$

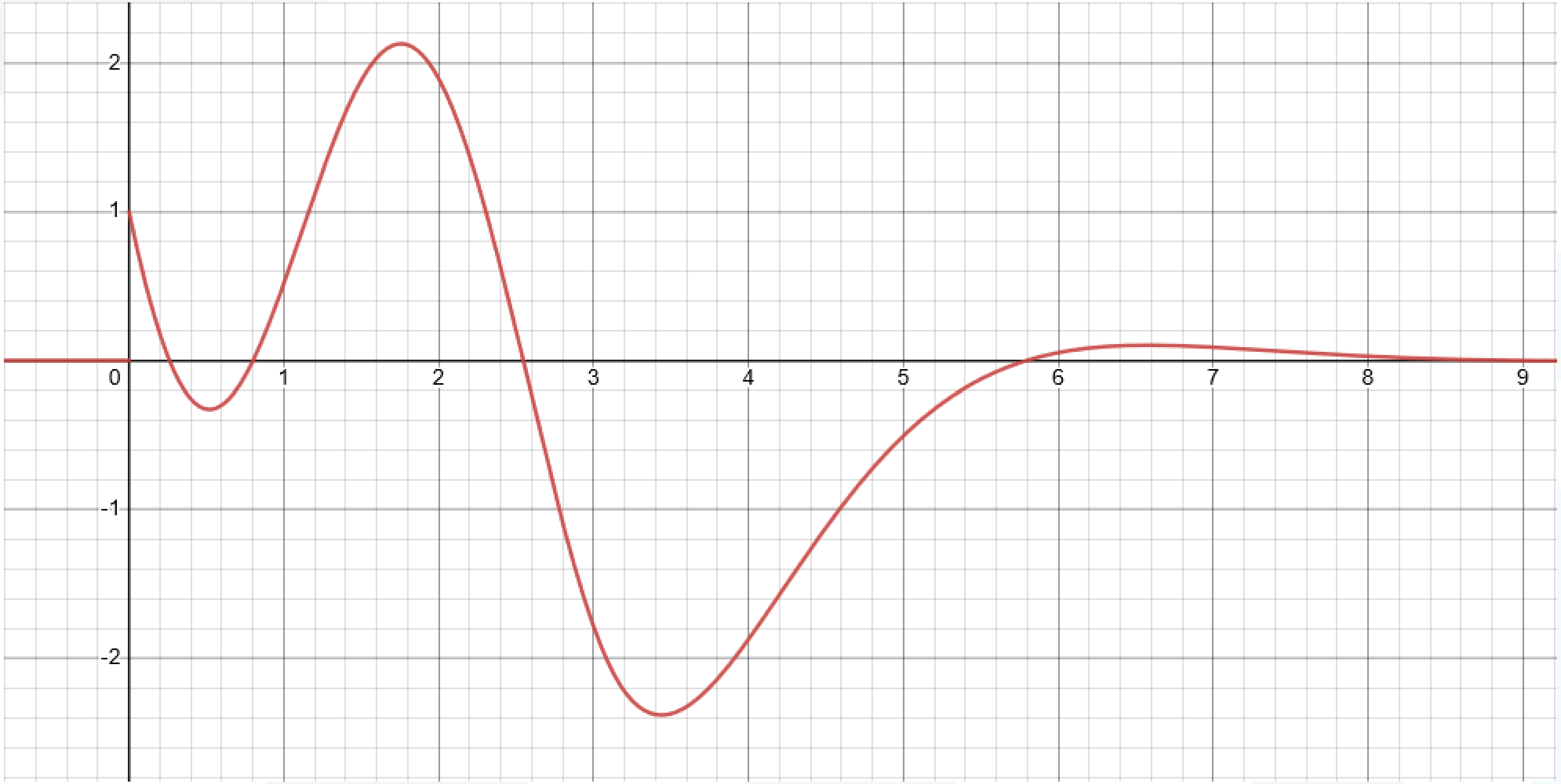
$$\mathcal{L}^{-1}\left\{-\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right\} = 2\cos(2t-2\pi) + \sin(2t-2\pi)$$

$$- e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$

$$\mathcal{L}^{-1}\left\{-\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right\} = 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t)$$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t + 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t)$$

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t], \qquad t > \pi$$


2
1
0
-1
-2
1
2
3
4
5
6
7
8
9

- **Theorem 12 (Initial Value Theorem):** If $\mathcal{L}\{f(t)\} = F(s)$ and if the indicated limits exist, then:

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

- **Example 23:** Initial value theorem

Calculate the initial value of the function $f(t)$, whose Laplace transform is:

$$F(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2 + 5^2}\right\} = 2e^{-t}\cos(5t) \Rightarrow f(0^+) = f(0) = 2$$

- Theorem 13 (Final Value Theorem): If $\mathcal{L}\{f(t)\} = F(s)$ and if the indicated limits exist, then:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- Example 24: Final value theorem

Calculate the final value of the function $f(t)$, whose Laplace transform is:

$$F(s) = \frac{s+3}{s(s+1)}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s(s+3)}{s(s+1)} = \lim_{s\to 0} \frac{s+3}{s+1} = 3$$

Verification:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{s(s+1)}\right\} = (3 - 2e^{-t}) \Rightarrow \lim_{t\to\infty} f(t) = 3$$

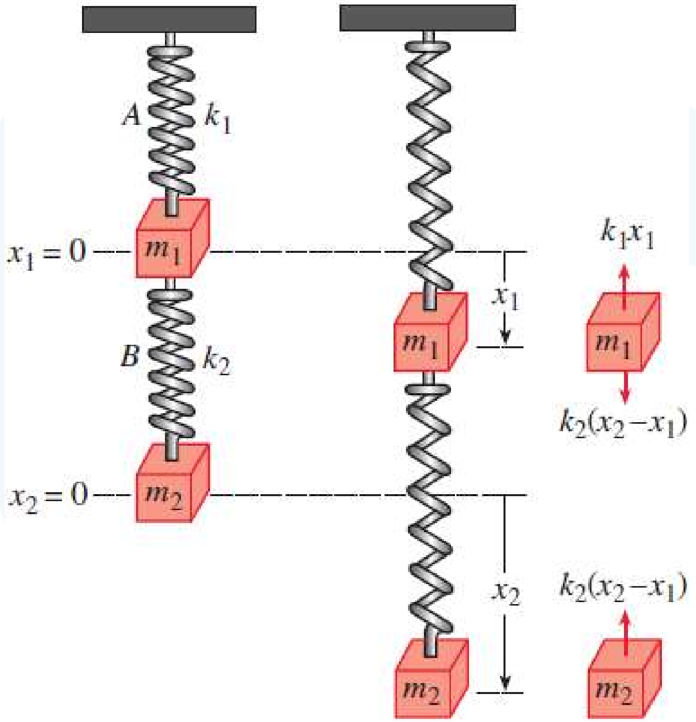
5. Systems of Linear Differential Equations

- When initial conditions are specified, the LT reduces a system of linear DEs with constant coefficients to a set of simultaneous algebraic equations in the transformed functions.

Coupled Springs

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1)$$

that describes the motion of two masses m_1 and m_2 in the coupled spring/mass system shown in the figure beside.



Equilibrium *Motion* *Forces*

- Example 25: Coupled Springs

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

$(k_1 = 6,\ k_2 = 4, m_1 = 1$, and $m_2 = 1)$

subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$

$$s^2 X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$
$$-4X_1(s) + s^2 X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0$$

$$(s^2 + 10)X_1(s) - 4X_2(s) = 1$$
$$-4X_1(s) + (s^2 + 4)X_2(s) = -1$$

Solving for $X_1(s)$ and using partial fractions on the result yields:

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

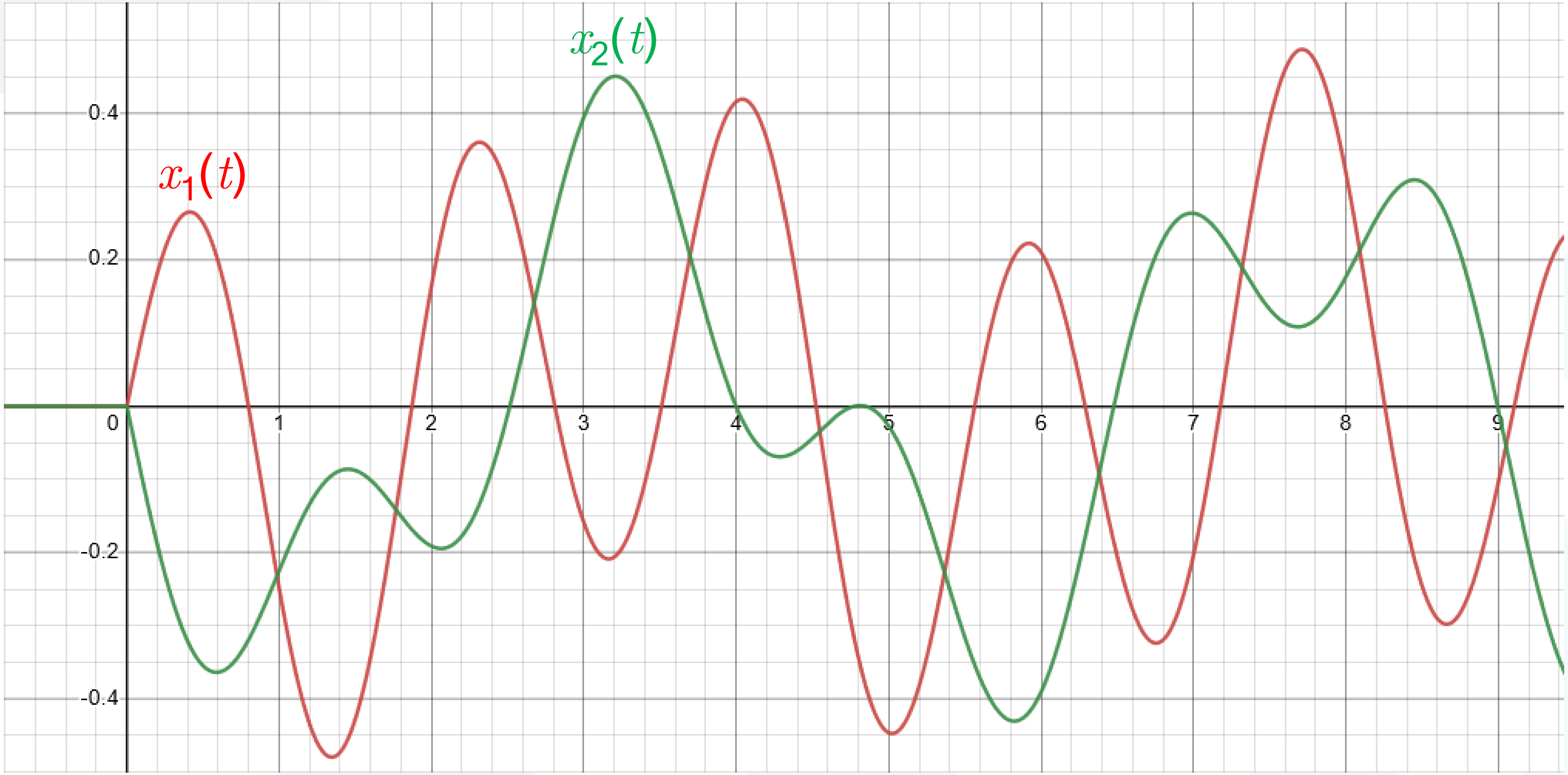
$$x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

Substituting the expression for $X_1(s)$ into the first equation

$$X_2(s) = \frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$$

The motion of each mass is harmonic (the system is undamped!), being the superposition of a slow oscillation and a rapid oscillation.

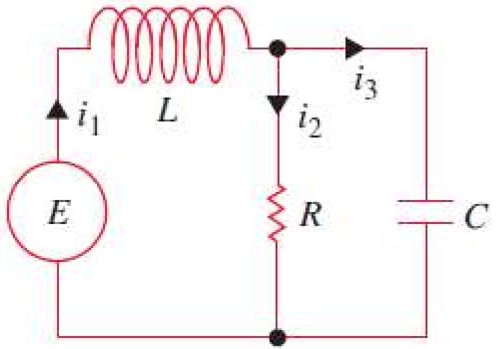


$x_1(t)$
$x_2(t)$
0.4
0.2
-0.2
-0.4
0
1
2
3
4
5
6
7
8
9

Networks

$$L\frac{di_1(t)}{dt} + Ri_2(t) = E(t)$$

$$RC\frac{di_2(t)}{dt} + i_2(t) - i_1(t) = 0$$



- Example 26: An Electrical Network

  Solve the system before under the conditions $E(t) = 60$ V, $L = 1$ H, $R = 50\ \Omega$, $C = 10^{-4}$ F, and the currents i_1 and i_2 are initially zero.

$$\frac{di_1(t)}{dt} + 50i_2(t) = 60$$

$$5\times10^{-3}\frac{di_2(t)}{dt} + i_2(t) - i_1(t) = 0$$

$$i_1(0) = i_2(0) = 0$$

$$sI_1(s) + 50I_2(s) = \frac{60}{s}$$

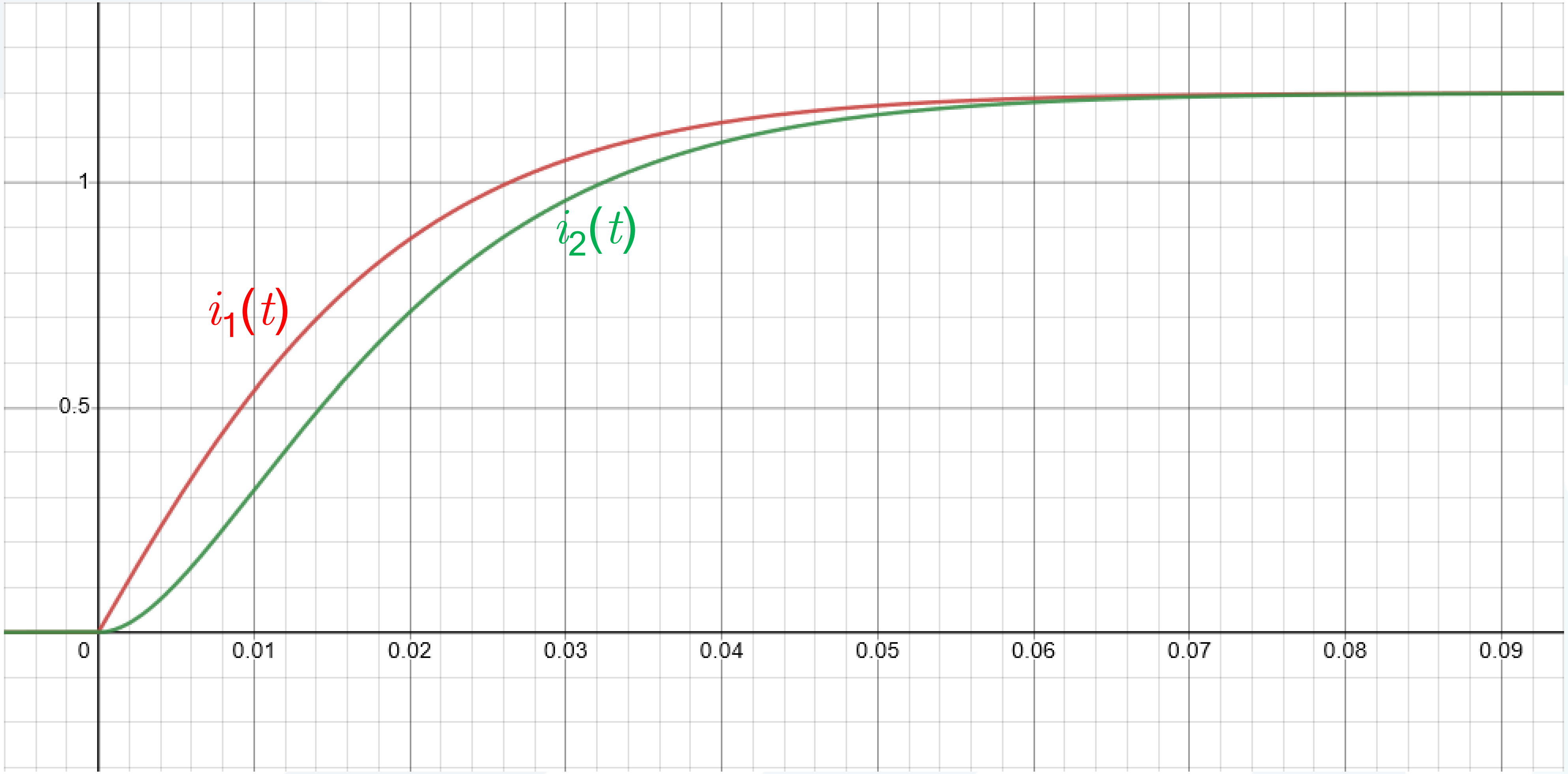
$$-200I_1(s) + (s+200)I_2(s) = 0$$

$$I_1(s) = \frac{60s + 12000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2}$$

$$i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}$$

$$I_2(s) = \frac{12000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}$$

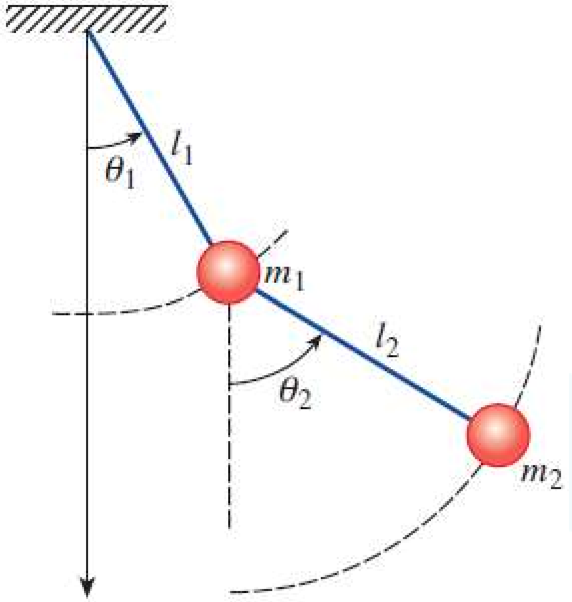
$$i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}$$


$i_1(t)$
$i_2(t)$
1
0.5
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09

Double Pendulum

$$(m_1 + m_2) l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2) l_1 g \theta_1 = 0$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g \theta_2 = 0$$

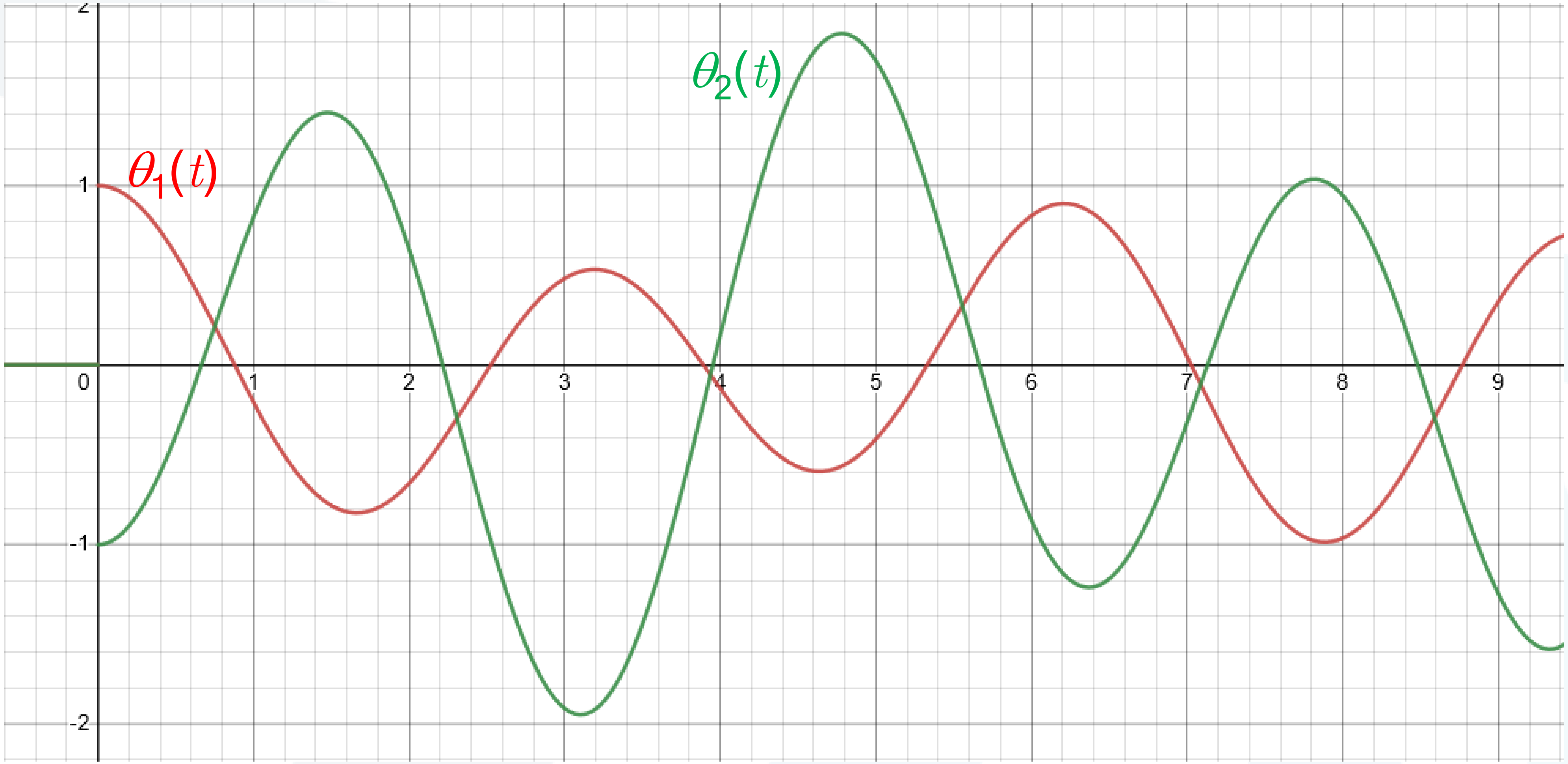


- **Example 27:** Double Pendulum

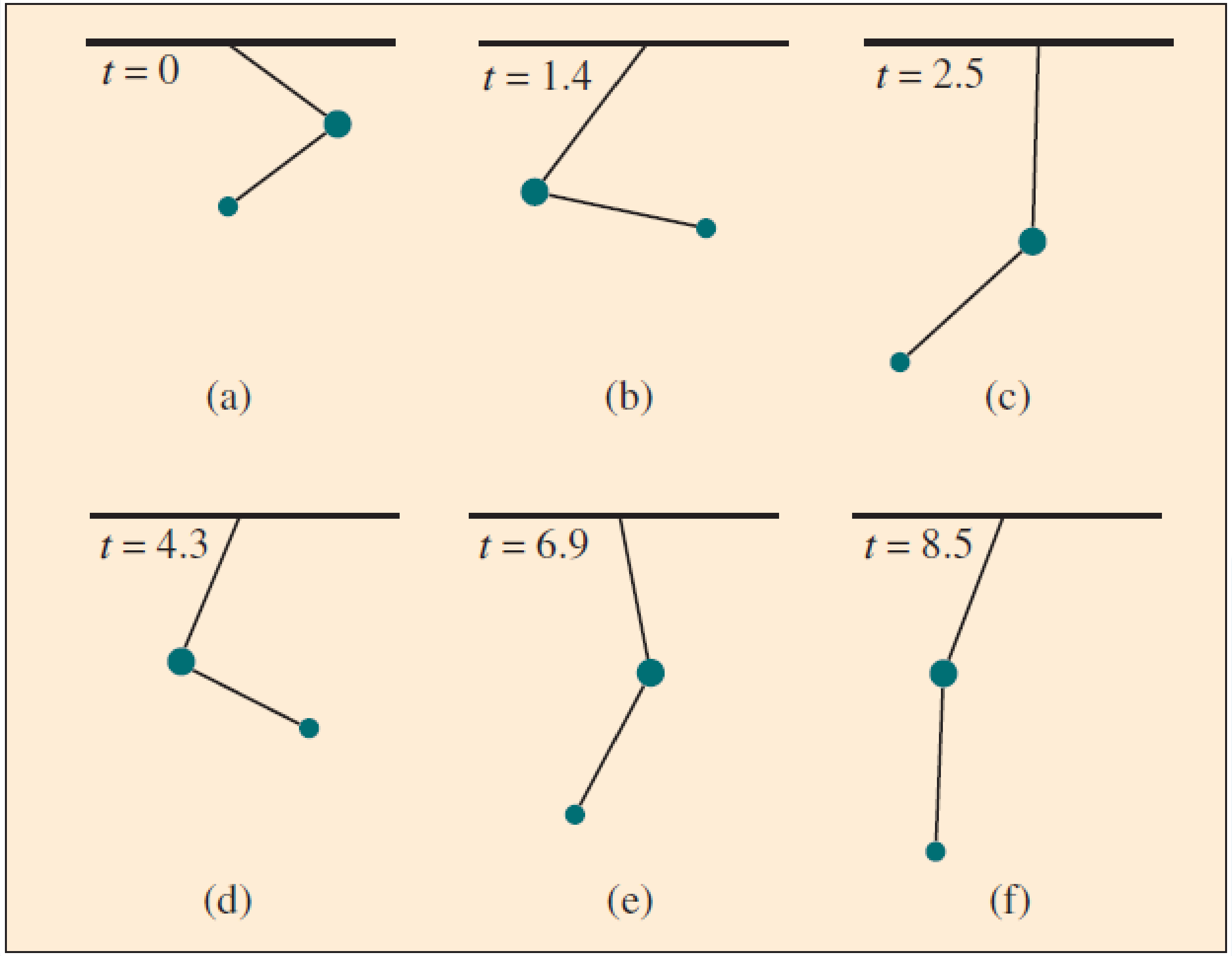
$m_1 = 3,\ m_2 = 1,\ l_1 = l_2 = 16,\ \theta_1(0) = 1,\ \theta_2(0) = -1,\ \theta_1'(0) = \theta_2'(0) = 0$

$$\theta_1(t) = \frac{1}{4}\cos\frac{2}{\sqrt{3}}t + \frac{3}{4}\cos 2t$$

$$\theta_2(t) = \frac{1}{2}\cos\frac{2}{\sqrt{3}}t - \frac{3}{2}\cos 2t$$



$\theta_1(t)$
$\theta_2(t)$
0
1
2
3
4
5
6
7
8
9
2
1
-1
-2



Positions of masses at various times

Solving Systems of Equations in Matrix Form

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}(t)$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

subject to the initial conditions $x_1(0) = x_1$, $x_2(0) = x_2$, …, $x_n(0) = x_n$

$$\boldsymbol{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}, \quad \boldsymbol{B}(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \\ \vdots \\ B_n(s) \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$sX(s) - x_0 = AX(s) + B(s)$$

$$[sI - A]X(s) = x_0 + B(s)$$

$$X(s) = [sI - A]^{-1}[x_0 + B(s)]$$

$$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}[x_0 + B(s)]\}$$

- Example 28: Initial value problem
  Solve the initial value problem

$$x' - 2x + y = \sin t$$
$$y' + 2x - y = 1$$

subject to $x(0) = 1,\ y(0) = -1$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B(s) = \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix}$$

$$\boldsymbol{X}(s) = \left[s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \right]^{-1} \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix} \right]$$

$$\boldsymbol{X}(s) = \begin{bmatrix} s-2 & 1 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} (s^2+2)/(s^2+1) \\ (1-s)/s \end{bmatrix} =$$

$$\boldsymbol{X}(s) = \begin{bmatrix} \dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)} \\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)} \end{bmatrix} \begin{bmatrix} \dfrac{s^2+2}{(s^2+1)} \\ \dfrac{1-s}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \\ \dfrac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)} \end{bmatrix}$$

$$x(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$$

$$y(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t}$$

Determination of e^{tA} by Means of the Laplace Transform

- The solution of the homogeneous system of equations: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ subject to the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, can be written

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0$$

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}\}\boldsymbol{x}_0$$

$$\Rightarrow e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$$

- **Theorem 14 (Determination of e^{tA} by Means of the Laplace Transform):** Let A be a real $n \times n$ matrix with constant elements. Then the exponential matrix

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$$

- Note: Theorem 14 determines e^{tA} in the cases when A is diagonalizable with real eigenvalues, when it is diagonalizable with complex conjugate eigenvalues, and also when it is not diagonalizable.

- Example 29: Determination of e^{tA}

$$A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$$

Matrix A has the distinct eigenvalues 1 and 2, and so is diagonalizable.

$$[sI - A] = \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \dfrac{s-5}{s^2 - 3s + 2} & \dfrac{6}{s^2 - 3s + 2} \\ \dfrac{-2}{s^2 - 3s + 2} & \dfrac{s+2}{s^2 - 3s + 2} \end{bmatrix}$$

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} 4e^{t} - 3e^{2t} & -6e^{t} + 6e^{2t} \\ 2e^{t} - 2e^{2t} & -3e^{t} + 4e^{2t} \end{bmatrix}$$

- Example 30: Determination of e^{tA}

$$A = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}$$

Matrix A has the complex conjugate eigenvalues $-1 \pm 2i$.

$$[sI - A] = \begin{bmatrix} s+3 & 4 \\ -2 & s-1 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \dfrac{s-1}{s^2+2s+5} & \dfrac{-4}{s^2+2s+5} \\ \dfrac{2}{s^2+2s+5} & \dfrac{s+3}{s^2+2s+5} \end{bmatrix}$$

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix}$$

- Example 31: Determination of e^{tA}

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Matrix A has the repeated eigenvalue 4 and is not diagonalizable.

$$[sI - A] = \begin{bmatrix} s-4 & -1 \\ 0 & s-4 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \frac{1}{s-4} & \frac{1}{(s-4)^2} \\ 0 & \frac{1}{s-4} \end{bmatrix} \Rightarrow e^{tA} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$$